PHY209 Electromagnetism
Assignment 7
Handed out: November 3, 2019

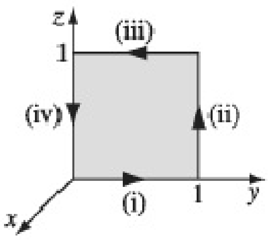
Problem 1

Consider two concentric spherical shells, of radii a and b. Suppose the inner one carries a charge q, and the outer one a charge $-q$ (both of them uniformly distributed over the surface). Calculate the energy of this configuration in two different ways:
(a) Use $W = \frac{\epsilon_0}{2}\int E^2 d\tau$.
(b) Use $W = \frac{\epsilon_0}{2}\left(\int E_1^2 d\tau + \int E_2^2 d\tau + 2\int \boldsymbol{E_1}\cdot\boldsymbol{E_2} d\tau\right)$.

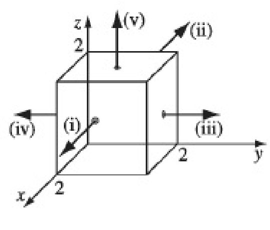
Problem 2

Find the interaction energy ($\epsilon_0 \int \boldsymbol{E_1}\cdot\boldsymbol{E_2} d\tau$) for two point charges, q_1 and q_2, a distance a apart. [Hint: Put q_1 at the origin and q_2 on the z axis; use spherical coordinates, and do the r integral first.]

Problem 3



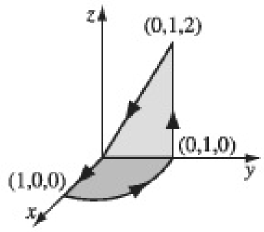
Suppose $\boldsymbol{v} = (2xz + 3y^2)\hat{\boldsymbol{y}} + (4yz^2)\hat{\boldsymbol{z}}$. Check Stokes' theorem for the square surface shown in Figure. Check that $\int(\nabla\times\boldsymbol{v})\cdot d\boldsymbol{a}$ depends only on the boundary line, not on the particular surface used by integrating over the five faces of the cube in Figure below. The back of the cube is open.



Problem 4

Compute the line integral of

$$\boldsymbol{v} = (r\cos^2\theta)\hat{\boldsymbol{r}} - (r\cos\theta\sin\theta)\hat{\boldsymbol{\theta}} + 3r\hat{\boldsymbol{\phi}} \quad (1)$$



around the path shown in Figure (the points are labeled by their Cartesian coordinates). Check your answer, using Stokes' theorem.

Problem 5

Determine the z component of the curl in cylindrical coordinates by explicitly calculating the cross product

$$\nabla \times \boldsymbol{A} = \left(\hat{\boldsymbol{\rho}} \frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{\boldsymbol{z}} \frac{\partial}{\partial z} \right) \times \left(A_\rho \hat{\boldsymbol{\rho}} + A_\phi \hat{\boldsymbol{\phi}} + A_z \hat{\boldsymbol{z}} \right) \tag{2}$$

Problem 6

A hydrogen atom (with the Bohr radius of half an angstrom) is situated between two metal plates 1 mm apart, which are connected to opposite terminals of a 500 V battery. What fraction of the atomic radius does the separation distance d amount to, roughly?

Problem 7

In Figure, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are (perfect) dipoles a distance r apart.
(a) Calculate the force on $\boldsymbol{p}_2$ due to $\boldsymbol{p}_1$ and the force on $\boldsymbol{p}_1$ due to $\boldsymbol{p}_2$. Are the answers consistent with Newton's third law?
(b) What is the torque on $\boldsymbol{p}_1$ due to $\boldsymbol{p}_2$? What is the torque on $\boldsymbol{p}_2$ due to $\boldsymbol{p}_1$? In each case, the torque on the dipole has to be evaluated about its own center.
(c) Find the total torque on $\boldsymbol{p}_2$ with respect to the center of $\boldsymbol{p}_1$ and compare it with the torque on $\boldsymbol{p}_1$ about that same point.

Problem 8

A dipole $\boldsymbol{p}$ is a distance r from a point charge q, and oriented so that $\boldsymbol{p}$ makes an angle θ with the vector $\boldsymbol{r}$ from q to $\boldsymbol{p}$.
(a) What is the force on $\boldsymbol{p}$?
(b) What is the force on q?

Problem 9

Evaluate the expressions using Einstein summation convention wherever possible.
(a) Prove that

$$[\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})] + [\boldsymbol{B} \times (\boldsymbol{C} \times \boldsymbol{A})] + [\boldsymbol{C} \times (\boldsymbol{A} \times \boldsymbol{B})] = 0 \tag{3}$$

Under what conditions does $[\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})] = -[\boldsymbol{C} \times (\boldsymbol{A} \times \boldsymbol{B})]$?
(b) Prove product rule

$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B}) \tag{4}$$

(c) Prove product rule

$$\nabla \times (f\boldsymbol{A}) = f(\nabla \times \boldsymbol{A}) - \boldsymbol{A} \times (\nabla f) \tag{5}$$

(d) Prove product rule

$$[\nabla \times (\boldsymbol{A} \times \boldsymbol{B})] = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) \tag{6}$$

(e) Prove product rule

$$[\nabla \cdot (\boldsymbol{A} \cdot \boldsymbol{B})] = [\boldsymbol{A} \times (\nabla \times \boldsymbol{B})] + [\boldsymbol{B} \times (\nabla \times \boldsymbol{A})] + \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) + \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) \tag{7}$$

Problem 10

(a) Compute $(\hat{\boldsymbol{r}} \cdot \nabla)\hat{\boldsymbol{r}}$, where $\hat{\boldsymbol{r}}$ is the unit vector.
(b) Prove that the divergence of a curl of a vector field is always zero.
(c) Prove that the curl of a gradient of a scalar field is always zero.
(d) Let $\boldsymbol{F}_1 = x^2\hat{\boldsymbol{z}}$ and $\boldsymbol{F}_2 = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$. Calculate the divergence and curl of both $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$. Which one can be written as the gradient of a scalar? Find a scalar potential that does the job. Which one can be written as the curl of a vector? Find a suitable vector potential.
(e) Show that

$$\int_{\mathcal{V}} (\nabla \times \boldsymbol{v}) d\tau = \oint_S d\boldsymbol{a} \times \boldsymbol{v} \tag{8}$$

[Hint: Replace $\boldsymbol{v}$ by $(\boldsymbol{v} \times \boldsymbol{c})$ in the divergence theorem.] Using this corrolary, show $\nabla \times (r^n \hat{\boldsymbol{r}}) = \boldsymbol{0}$.

Problem 11

(a) Show that

$$\int_{\mathcal{V}} (\nabla T) d\tau = \oint_S T d\boldsymbol{a} \tag{9}$$

[Hint: Let $\boldsymbol{v} = \boldsymbol{c}T$, where $\boldsymbol{c}$ is a constant vector, in the divergence theorem].
(b) The integral $\boldsymbol{a} = \int_S d\boldsymbol{a}$ is called the vector area of the surface S. Show that $\boldsymbol{a} = \boldsymbol{0}$ for any closed surface. [Hint: Use the result from (a)].
(c) Show that $\boldsymbol{a}$ is the same for all surfaces sharing the same boundary.
(d) Show that

$$\int_S \nabla T \times d\boldsymbol{a} = -\oint_C T d\boldsymbol{l} \tag{10}$$

[Hint: Let $\boldsymbol{v} = \boldsymbol{c}T$ in Stokes theorem.]
(e) Show that

$$\oint \boldsymbol{c} \cdot \boldsymbol{r} d\boldsymbol{l} = \boldsymbol{a} \times \boldsymbol{c}, \tag{11}$$

for any constant vector $\boldsymbol{c}$. [Hint: Let $T = \boldsymbol{c} \cdot \boldsymbol{r}$ in (d).]